Koide's Z_3-symmetric parametrization, quark masses and mixings

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Abstract

The sets of charged-lepton (L) and quark (D, U) masses may be parametrized in a Z_3-symmetric language appropriate for the discussion of Koide's formula. Experiment suggests that at the low-energy scale the relevant phase parameters δ_f take on possibly exact values of $\delta_L = 3\delta_D/2 = 3\delta_U = 2/9$. For k_f (the other parameter relevant for the pattern of masses), a similarly simple expression ($k_L = 1$) is known for charged leptons only. Using the Fritzsch-Xing decomposition of quark-mixing matrices, we show that the suggested pattern of low-energy quark masses is consistent with an earlier conjecture that $k_{D,U} \approx 1$ in the weak basis.

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1 The doubly special Koide's parametrization

The appearance of three generations of leptons and quarks and the related issue of their masses started to baffle us three quarters of a century ago. Over the years the problem has become further complicated by the presence of inter-generation mixing, as revealed in weak interactions. Fortunately, various approximate regularities have been found in the observed pattern of particle masses and mixings. Among the many possible parametrizations of these regularities, there might be some whose simplicity could help us in deciphering physics beyond the Standard Model.

One of the most interesting of such regularities is an empirical relation between the charged-lepton masses discovered by Koide [1] (for a brief review see [2]):

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{1 + k_L^2}{3}, \tag{1}$$

with k_L equal exactly 1. When the experimental e and μ masses (here taken from [3]) are inserted into Eq. (1), this relation predicts the tauon mass within one standard deviation from its observed value:

$$\begin{aligned} m_\tau(k_L = 1) &= 1776.9689 \text{ MeV} \qquad (2)\\ m_\tau(\exp) &= 1776.82 \pm 0.16 \text{ MeV}. \qquad (3) \end{aligned}$$

Discussions of this success of Koide's formula (1) are naturally formulated in a Z_3-symmetric framework by parametrizing the masses of any three given fermions f_1, f_2, f_3 in terms of three parameters M_f, k_f, δ_f as [4, 5]:

$$\sqrt{m_{f_j}} = \sqrt{M_f}\left(1 + \sqrt{2}\, k_f \cos\left(\frac{2\pi j}{3} + \delta_f\right)\right), \qquad (j = 1, 2, 3). \tag{4}$$

This choice of parametrization of masses is particularly suited to Koide's formula as not only M_f but also δ_f drop out of the r.h.s. of Eq. (1).

Since δ_f is free we may assume $m_1 \leq m_2 \leq m_3$ without any loss of generality. From Eq. (4) one then gets a counterpart of Eq. (1), in which it is now k_L that drops out of the formula:

$$\frac{\sqrt{3}\,(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}} = \tan \delta_L. \tag{5}$$

From the experimental values of e, μ and τ masses one finds:

$$\delta_L = 0.2222324, \tag{6}$$

which, as observed by Brannen and Rosen [6, 7], is extremely close to

$$\delta_L = 2/9. \tag{7}$$

Conversely, assuming $\delta_L = 2/9$, Eq. (5) predicts the value of the tauon mass in terms of experimental e and μ masses. Just as in the case of Koide's formula, the relevant prediction is within one standard deviation from the measured τ mass:

$$m_\tau(\delta_{\rm L} = 2/9) = 1776.9664 \text{ MeV}. \tag{8}$$

Assuming that the Koide and the Brannen-Rosen observations do not reflect mere coincidences, the Z_3-symmetric parametrization (4) should be rightly called 'doubly special'. A peculiar feature of this parametrization is that the simple numbers of 1 and 2/9 work well at the *low*-energy scale and not at some high mass scale. For example, taking the values of charged-lepton masses at the mass scale of M_Z, the extracted values of k_L and δ_L deviate from their 'perfect' values of 1 and 2/9 by about 0.2 % and 0.5 % respectively. Apparently, an explanation of the success of predictions (2, 8) should not be sought at the high-mass scale of some grand unified theory (see eg. [8]).

2 Extending the scheme to the quark sector

If there is some physical reason behind the appearance of simple numbers such as 1 and 2/9 in the charged-lepton sector, one would expect its analogs working in the quark and neutrino sectors as well. However, it is known that the original Koide formula (1) does not work when replacing the charged-lepton masses with those of neutrinos or quarks. For neutrinos one estimates directly from experiment that $k_\nu \leq 0.81$ [9] (the mathematically allowed region being $0 \leq k_f \leq \sqrt{2}$). For quarks, using their mass values appropriate at $\mu = 2\ GeV$, one obtains $k_D \approx 1.08$ ($k_U \approx 1.25$) for the down (up) quarks respectively [9, 10]. If a higher energy scale $\mu = M_Z$ is taken, even larger values are obtained, i.e. $k_D = 1.12$ and $k_U = 1.29$. Going from $\mu = 2\ GeV$ towards the low energy scale leads to smaller values of k_D and k_U. However, the top quark mass is so large that one certainly cannot bring k_U into the vicinity of 1.

On the other hand, it has been observed recently [11] from the quark sector analogs of Eq. (5) that at the *low*-energy scale the relevant phase parameters acquire approximate values:

$$\begin{aligned} \delta_U &\approx 2/27 = \delta_L/3, \\ \delta_D &\approx 4/27 = 2\delta_L/3. \end{aligned} \tag{9}$$

Due to the problem of quark confinement we obviously cannot check how precise the above equalities are. However, given the accuracy of their lepton counterpart (Eq. (8)) we may expect that they are nearly exact. Therefore, we will assume this from now on and *conjecture* that at the low energy scale the charged-lepton and quark mass bases are characterised by Eqs. (7, 9).

The problem then remains how to interpret the value of $k_L = 1$ and how it should be generalized to the quark sector. Here we accept the suggestion of ref. [12] that $k_f = 1$ is a feature of the weak basis. Thus, according to [12], the masses of charged leptons are described by $k_L = 1$ because for charged leptons the mass and the weak bases coincide. The lepton mixing matrix (for simplicity we assume here Dirac neutrino masses), i.e.

$$V_{MNS} = U_L^\dagger U_\nu, \tag{10}$$

is then wholly assigned to the contribution from neutrinos (for which $k_\nu \neq 1$):

$$\begin{aligned} U_\nu &= V_{MNS}, \\ U_L &= 1. \end{aligned} \tag{11}$$

In other words, if U_L were different from 1, one would not expect the simplicity of Koide's formula to persist.

Since the analogs of the charged-lepton equality $k_L = 1$ do not hold in the U and D quark sectors, one expects that the quark counterpart of the decomposition (11) will be also modified. Thus, in the CKM matrix

$$V_{CKM} = U_U^\dagger U_D, \tag{12}$$

both of the factor matrices U_U and U_D are expected to be different from 1.
For fermion type f the general connection between the mass and the weak bases is

$$\mathrm{diag}(m_{f_1}, m_{f_2}, m_{f_3}) = U^\dagger_{f,left} M U_{f,right}, \tag{13}$$

where M is the mass matrix in the weak basis. In the above formula $U_{f,left} \equiv U_f$, while $U_{f,right}$ may be chosen equal to 1. As a result, in the weak basis one deals with 'pseudo-masses' $\tilde{m}_{f_j}$ defined as [12]

$$\tilde{m}_{f_j} = |\sum_l U_f^{jk} m_{f_k}|. \tag{14}$$

For charged leptons ($f = L$, $U_L = 1$) these pseudo-masses coincide with the observed masses, i.e. $\tilde{m}_{L_j} = |m_{L_j}|$. On the other hand, for quarks ($f = D, U$ with $U_D, U_U \neq 1$) the pseudo-masses are different from the observed mass values. It is for these 'pseudo-masses' that, according to the proposal of [12], the analogs of Koide's formula (1) are supposed to hold with $k_D = k_U = 1$. The authors of ref. [12] used quark masses at the Z mass scale and found that it is possible to get $k_D = k_U \approx 1$ provided one takes a value of the strange quark mass $m_s(Z)$ that is larger by a factor of 2.5 (!) from the theoretical estimate at that scale. Since the Koide and the Brannen-Rosen observations work best at the *low*-energy scale (and not at the Z mass), and since at the low-energy scale the masses of quarks (and especially m_s, see [11]) are naturally expected to be larger than at the Z mass, a question appears if it is possible to recover Koide's formula for quarks using low-energy quark masses corresponding to phases of formulas (9). This is the question asked here. Thus, the present paper constitutes a low-energy-scale study of the idea of ref.[12].

3 The structure of U_f in the quark sector

As the assumption of Koide's formula for pseudo-masses imposes constraints upon matrices U_D and U_U (and, consequently, upon V_{CKM}), we have to discuss these matrices in some detail.

In [13] Fritzsch and Xing convincingly argued that the hierarchical structure of quark mass terms suggests certain particular parametrizations of U_D and U_U as 'most physical' (i.e. that it selects one of the nine possible parametrizations of the V_{CKM} matrix [14] as probably the most suitable for the description of the quark-mixing phenomenon). The same parametrization was also advocated in ref. [12] where the conjecture that Koide's formula holds in the weak basis was originally formulated. Consequently, we think that it is justified to accept the Fritzsch-Xing parametrization here. The relevant 'natural' parametrizations of U_D and U_U are then:

$$\begin{aligned} U_D &= R_{23}(\phi_b, \theta_b)\, R_{12}(\theta_d), \\ U_U &= R_{23}(\phi_t, \theta_t)\, R_{12}(\theta_u), \end{aligned} \tag{15}$$

with ($c_q \equiv \cos\theta_q,\ s_q \equiv \sin\theta_q$)

$$R_{12}(\theta_q) = \begin{pmatrix} c_q & -s_q & 0 \\ s_q & c_q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (q = d, u), \tag{16}$$

$$R_{23}(\phi_q, \theta_q) = \begin{pmatrix} e^{-i\phi_q} & 0 & 0 \\ 0 & c_q & s_q \\ 0 & -s_q & c_q \end{pmatrix}, \qquad (q = b, t). \tag{17}$$

Thus, the induced parametrization of the V_{CKM} matrix can be read from:

$$V_{CKM} = R_{12}^\dagger(\theta_u)\, R_{23}^\dagger(\phi_t, \theta_t)\, R_{23}(\phi_b, \theta_b)\, R_{12}(\theta_d). \tag{18}$$

The product $R_{23}^\dagger(\phi_t, \theta_t) R_{23}(\phi_b, \theta_b)$ may be written in the form of Eq. (17) with a single phase $\phi = \phi_b - \phi_t$ and a single rotation angle $\theta = \theta_b - \theta_t$, (with $c \equiv \cos\theta$, $s \equiv \sin\theta$)

$$R_{23}^\dagger(\phi_t, \theta_t) R_{23}(\phi_b, \theta_b) \equiv R_{23}(\phi, \theta) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}. \tag{19}$$

The CKM matrix is then parametrized as

$$\begin{aligned} V_{CKM} &= \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} s_u s_d c + c_u c_d e^{-i\phi} & s_u c_d c - c_u s_d e^{-i\phi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\phi} & c_u c_d c + s_u s_d e^{-i\phi} & c_u s \\ -\, s_d s & -\, c_d s & c \end{pmatrix}. \end{aligned} \tag{20}$$

Since quark fields can be freely rephased, all of the three angles $\theta_d, \theta_u, \theta$ can be arranged to lie in the first quadrant so that s_u, s_d, s and c_u, c_d, c are all positive (the phase ϕ cannot be so restricted).

We use the following absolute values of the elements of the CKM matrix most relevant for our parametrization [3]:

$$\begin{aligned}
|V_{ub}| &= 0.00351 \pm 0.00015, \\
|V_{cb}| &= 0.0412 \pm 0.0008, \\
|V_{td}| &= 0.00867 \pm 0.00030, \\
|V_{ts}| &= 0.0404 \pm 0.0008, \\
|V_{tb}| &= 0.999146 \pm 0.000034.
\end{aligned} \tag{21}$$

Inserting the above numbers, we find from (20):

$$\begin{aligned}
\theta_u &= 4.87^\circ \pm 0.23^\circ, \\
\theta_d &= 12.11^\circ \pm 0.47^\circ, \\
\theta = \theta_b - \theta_t &= 2.37^\circ \pm 0.05^\circ.
\end{aligned} \tag{22}$$

Since formula (14) involves absolute values, the actual sizes of ϕ_b and ϕ_t and, consequently, the experimentally imposed restriction on the CP-violating phase parameter $\phi = \phi_b - \phi_t$ are irrelevant for our purposes.

4 Imposing Koide's condition on pseudo-masses

The values of $\delta_D = 4/27$ and $\delta_U = 2/27$, together with the low-energy ratios $m_s/m_d = 20.4$, $m_u/m_d = 0.56$ (see e.g. [15]) suffice to fix the pattern of low-energy quark masses up to two overall mass scales (in the up and down sectors). These scales are irrelevant for the discussion of Koide's formulas for pseudo-masses. For illustrative purposes, however, one may set $m_s = 160.0\ MeV$ and $m_t = 172000\ MeV$. This choice leads to the following representative values of *low*-energy quark masses (in MeV):

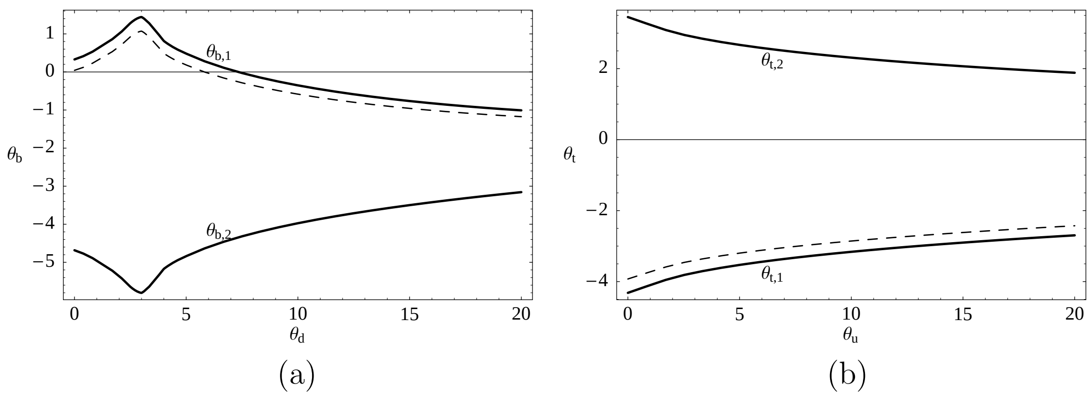
$$\begin{aligned}
&m_d = 7.843, \qquad m_s = 160.0, \qquad m_b = 4209, \\
&m_u = 4.392, \qquad m_c = 1296, \qquad m_t = 172000.
\end{aligned} \tag{23}$$

For a given value of k_f, with ϕ_b, ϕ_t dropping from expression (14) for the pseudo-masses, the condition

$$\frac{\sum_j \tilde{m}_{f_j}}{\left(\sum_j \sqrt{\tilde{m}_{f_j}}\right)^2} = \frac{1+k_f^2}{3} \tag{24}$$

imposes two constraints: between θ_d and θ_b in the down quark sector, and between θ_u and θ_t in the up quark sector. Thus, θ_b becomes dependent on θ_d (and θ_t on θ_u).

Figure 1: Correlations required by Eq. (24): (a) $\theta_b \leftrightarrow \theta_d$ and (b) $\theta_t \leftrightarrow \theta_u$ (all angles in degrees). Solid lines correspond to $k_D = k_U = 1$. Dashed lines denote solutions $\theta_{b,1}$ and $\theta_{t,1}$ for $k_D = k_U = 1.015$.



For $k_D = k_U = 1$ there are two possible solutions for function $\theta_b(\theta_d)$ and two possible solutions for function $\theta_t(\theta_u)$. They are shown in Fig. 1 with solid lines marked as $\theta_{b,n}$ and $\theta_{t,n}$ $(n = 1, 2)$. However, only the combination $\theta_{\rm Koide} = \theta_{b,1} - \theta_{t,1}$ is positive. Specifically, taking the central experimental values of θ_u and θ_d, one finds

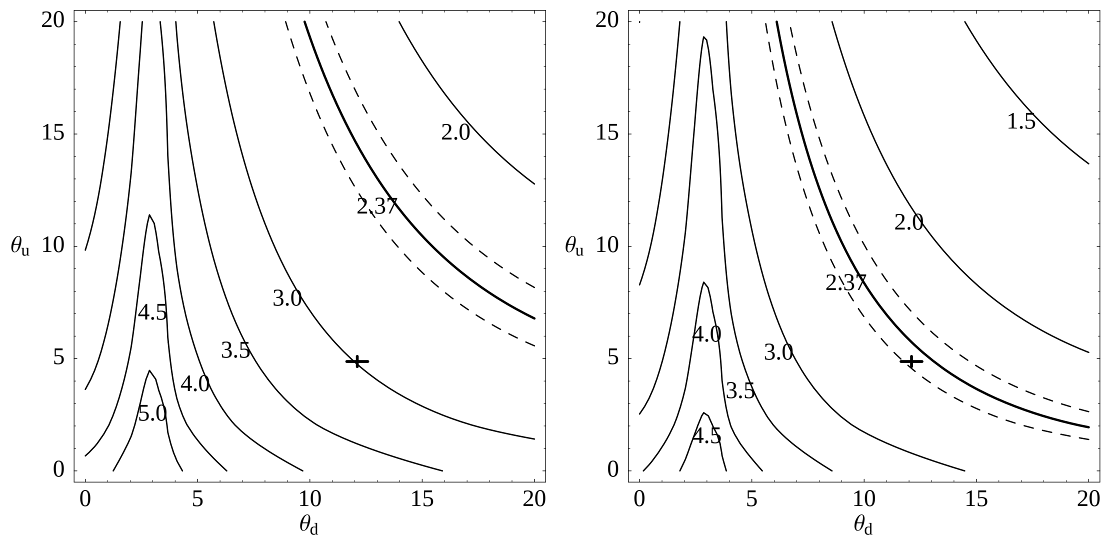
$$\theta_{\rm Koide} = 2.98^\circ, \tag{25}$$

which is only slightly different from the value given in Eq. (22).

A question thus emerges how big a departure of k_D, k_U from 1 is needed to fit the current experimental value of θ. The dashed lines in Fig. 1 correspond to $\theta_{b,1}$ and $\theta_{t,1}$ as obtained for $k_D = k_U = 1.015$. The predicted value of θ is then $\theta(k_D = k_U = 1.015) \approx 2.44^\circ$, which is in good agreement with experiment. For completeness, we also show the contour plot of θ as a function of θ_d and θ_u for $k_D = k_U = 1$ (Fig. 2a) and $k_D = k_U = 1.015$ (Fig. 2b). Dashed lines correspond to contours $2.37^\circ \pm 2\sigma$ (with $\sigma = 0.05^\circ$).

Although it might seem that the above results indicate that one cannot obtain $k_U = k_D = 1$, this is not the case. One has to remember that while parametrizations (15) have been suggested as the most appropriate ones [13], they may be 'naturally' modified. Indeed, the same V_{CKM} is obtained if one substitutes $U_D \to U'_D = W_D U_D$ and $U_U \to U'_U = W_U U_U$, provided W_D and W_U denote the same arbitrary unitary matrix. If $W_D \neq W_U$, but both W_D and W_U are very close to 1, a minor modification of our results is expected. Such a natural modification of $U_{D(U)} = R_{23}^{D(U)} R_{12}^{D(U)}$ is obtained if we put $W_{D(U)} = R_{13}^{D(U)}(\theta_{13}^{D(U)})$ with very small $\theta_{13}^D \neq \theta_{13}^U$. Since the inclusion of two new parameters $\theta_{13}^{D(U)}$ introduces additional freedom into the scheme, it does not make sense to study it here. However - keeping this freedom in mind - the parametrization of Eq.(15) and our numerical results may be viewed as capturing the dominant effects only.

Figure 2: Contour plots of $\theta = \theta_{b,1}(\theta_d) - \theta_{t,1}(\theta_u)$ (all angles in degrees): (a) $k_D = k_U = 1$ and (b) $k_D = k_U = 1.015$. Dashed lines are contours corresponding to $\theta = 2.37° \pm 0.10°$. Cross denotes experimental point $(\theta_d, \theta_u) = (12.11° \pm 0.47°, 4.87° \pm 0.23°)$.



(a) (b)

In conclusion, the data are consistent with the statement that low-energy quark masses satisfy phase relations $\delta_D = 2\delta_U = 4/27$, while the expected Koide relations $k_U = k_D = 1$ hold approximately for masses transformed to the weak basis, as suggested in [12]. These observations might be relevant for a future theory of mass.

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